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A collocation method for high-frequency scattering by convex polygons

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Abstract

We consider the problem of scattering of a time-harmonic acoustic incident plane wave by a sound soft convex polygon. For standard boundary or finite element methods, with a piecewise polynomial approximation space, the computational cost required to achieve a prescribed level of accuracy grows linearly with respect to the frequency of the incident wave. Recently Chandler–Wilde and Langdon proposed a novel Galerkin boundary element method for this problem for which, by incorporating the products of plane wave basis functions with piecewise polynomials supported on a graded mesh into the approximation space, they were able to demonstrate that the number of degrees of freedom required to achieve a prescribed level of accuracy grows only logarithmically with respect to the frequency. Here we propose a related collocation method, using the same approximation space, for which we demonstrate via numerical experiments a convergence rate identical to that achieved with the Galerkin scheme, but with a substantially reduced computational cost.

MSC: 35J05; 65N38; 65R20

Keywords: Collocation method; High-frequency scattering; Oscillatory integrals

1. Introduction

Consider the problem of scattering of a time-harmonic acoustic incident plane wave u^i by a sound soft convex polygon Ω. The total acoustic field u satisfies

$$\Delta u(\mathbf{x}) + k^2 u(\mathbf{x}) = 0, \quad \mathbf{x} \in D := \mathbb{R}^2 \backslash \bar{\Omega}, \tag{1}$$

$$u(\mathbf{x}) = 0, \quad \mathbf{x} \in \Gamma := \partial\Omega, \tag{2}$$

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where the wavenumber $k > 0$ is proportional to the frequency of the incident wave, together with the Sommerfeld radiation condition

$$\lim_{r\to\infty} r^{1/2}\left(\frac{\partial u^s}{\partial r} - \mathrm{i}ku^s\right) = 0, \tag{3}$$

on the scattered field $u^s := u - u^i$, where $r := |\mathbf{x}|$ and the limit holds uniformly in all directions $\mathbf{x}/|\mathbf{x}|$. Existence and uniqueness of a solution $u \in C(\overline{D}) \cap C^2(D)$ to (1)–(3) follows from classical results; see [8] for details.

Using Green's theorem we can represent $u(\mathbf{x})$, $\mathbf{x} \in D$, as a combination of single and double layer potentials, and with the double layer potential disappearing due to (2) we have [11, Theorem 3.12]

$$u(\mathbf{x}) = u^i(\mathbf{x}) - \int_\Gamma \Phi(\mathbf{x}, \mathbf{y})\frac{\partial u}{\partial \mathbf{n}}(\mathbf{y})\,\mathrm{d}s(\mathbf{y}), \quad \mathbf{x} \in D. \tag{4}$$

Here $\Phi(\mathbf{x}, \mathbf{y}) := (\mathrm{i}/4)H_0^{(1)}(k|\mathbf{x}-\mathbf{y}|)$ is the standard fundamental solution for the Helmholtz equation and $\mathbf{n}$ is the normal vector directed out of Ω. Thus our problem reduces to finding the complementary boundary data $\partial u/\partial \mathbf{n} \in L^2(\Gamma)$, and to do this we solve the well-known second kind integral equation

$$(I + K)\frac{\partial u}{\partial \mathbf{n}} = f \quad \text{on } \Gamma\backslash\{S\}, \tag{5}$$

where S is the set of corners of Ω, $f := 2\partial u^i/\partial \mathbf{n} + 2\mathrm{i}\eta u^i$, and for $v \in L^2(\Gamma)$

$$Kv(\mathbf{x}) := 2\int_\Gamma \left(\frac{\partial \Phi(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}(\mathbf{x})} + \mathrm{i}\eta\Phi(\mathbf{x}, \mathbf{y})\right) v(\mathbf{y})\,\mathrm{d}s(\mathbf{y}),$$

where η is a coupling parameter, with $\eta \in \mathbb{R}\backslash\{0\}$ ensuring that (5) has a unique solution (again we refer to [8] for details).

The formulae (4) and (5) also apply for non-polygonal obstacles. For any obstacle, the kernel, right-hand side, and solution of (5) all oscillate rapidly when k is large, and thus it is well known that the computational cost of solving (5) to a prescribed level of accuracy by standard schemes, with piecewise polynomial approximation spaces, grows at least linearly with respect to the wavenumber k (see e.g. [8,21] and the references therein). However, by removing the high-frequency asymptotics and solving a modified integral equation whose solution approaches zero almost everywhere as $k \to \infty$, it is possible to devise numerical schemes for solving integral equations such as (5) with computational costs that grow at a sublinear rate as k increases (see e.g. [1,6,8,19]).

In particular, in [8] Chandler-Wilde and Langdon recently proposed a novel Galerkin boundary element method for solving (5) in the case of a polygonal obstacle for which it was demonstrated via both a rigorous error analysis and numerical simulations that the number of degrees of freedom required to solve (5) (and thus (1)–(3)) to a prescribed level of accuracy grows only logarithmically with respect to k. This appears to be the best result to date for problems of scattering by bounded obstacles. It was achieved by removing the leading order high-frequency asymptotic behaviour from (5),

$$\frac{1}{k}\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}(s)) = \Psi(s) + \mathrm{e}^{\mathrm{i}ks}v_+(s) + \mathrm{e}^{-\mathrm{i}ks}v_-(s), \tag{6}$$

for $s \in [0, L]$ (where $\mathbf{x}(s)$, $s \in [0, L]$, parametrises Γ), with the leading order term $\Psi(s)$ given by (10) below. The function Ψ is known explicitly for any convex obstacle. However, for the particular case of a convex polygon, a consideration in [8] of a related set of half plane problems demonstrates that the functions $v_\pm$ and all their derivatives are highly peaked near the corners of the polygon, and rapidly decaying away from the corners. The oscillatory nature of $\partial u/\partial \mathbf{n}$ is thus represented exactly in (6) by the known leading order term and the terms $\mathrm{e}^{\pm\mathrm{i}ks}$, and to approximate $\partial u/\partial \mathbf{n}$ all that is required is to approximate the smooth functions $v_\pm$. These functions decay sufficiently quickly that the number of degrees of freedom required to maintain the accuracy of their best L^2 approximation from a space of piecewise polynomials supported on a graded mesh, with a higher concentration of mesh points closer to the corners of the polygon, grows only logarithmically with respect to k as $k \to \infty$.

The question then arises of how we might go about selecting our best L^2 approximation to $v_\pm$ from the approximation space. In [8] a Galerkin scheme is used, for which both stability and convergence are proved. However, the

implementation of this scheme requires the evaluation of many highly oscillatory double integrals, which can become computationally expensive at high frequencies. Although there has been some recent work on the efficient evaluation of highly oscillatory double integrals (see e.g. [13,15]) many questions remain unanswered. By contrast, several integration schemes have recently been proposed in the literature specifically for the evaluation of highly oscillatory single integrals, and many of these new schemes have the property that their performance actually improves as the integrand becomes more oscillatory (see for example [6,14,16,17] and the references therein). Using this as our motivation in the current paper, here we investigate the application of a collocation method for the solution of (5). We use the same approximation space, and thus we might anticipate achieving a similar sublinear convergence rate with respect to k as that achieved by the Galerkin scheme in [8], but the collocation scheme has the advantage that its implementation requires only the evaluation of highly oscillatory single integrals.

We begin in Section 2 by defining the approximation space more precisely, introducing our collocation method, and making some remarks about its conditioning, stability and convergence properties. We proceed in Section 3 with a full description of how the scheme is implemented, including a discussion of how we can evaluate the highly oscillatory single integrals which arise. In Section 4 we present some numerical results, demonstrating that the collocation method appears to converge to the same solution as the Galerkin scheme, for which a full error analysis has been carried out in [8], but with a significant reduction in computational cost. Finally in Section 5 we present some conclusions.

2. The boundary element method

We begin by defining some notation, as in Fig. 1, where the total field is shown for a problem of scattering by a square of side length 2π, with $k = 10$. We write the boundary of the polygon as $\Gamma = \bigcup_{j=1}^{n} \Gamma_j$, where Γ_j, $j = 1, \ldots, n$ are the n sides of the polygon, ordered so that Γ_j, $j = 1, \ldots, n_s$ are in shadow, and Γ_j, $j = n_s + 1, \ldots, n$ are illuminated. The shadow zone, below and to the right of the obstacle, and the interference pattern above Γ_3 and to the left of Γ_4 can clearly be seen in Fig. 1. We denote the corners of the polygon by $P_j := (p_j, q_j)$, $j = 1, \ldots, n$, and we set $P_{n+1} = P_1$, so that for $j = 1, \ldots, n$, Γ_j is the line joining P_j with P_{j+1}. We denote the length of Γ_j by $L_j := |P_{j+1} - P_j|$, the external angle at each vertice P_j by $\Omega_j \in (\pi, 2\pi)$, the outward normal to the line Γ_j by $\mathbf{n}_j := (n_{j1}, n_{j2})$, and the angle of the incident plane wave, as measured anticlockwise from the downward vertical, by $\theta \in [0, \pi/2]$. Writing $\mathbf{x} = (x_1, x_2)$ we then have $u^i(\mathbf{x}) = e^{ik(x_1 \sin\theta - x_2 \cos\theta)} = e^{ik\mathbf{x}.\mathbf{d}}$, where $\mathbf{d} := (\sin\theta, -\cos\theta)$. Defining further for $j = 1, \ldots, n$,

$$a_j := \frac{p_{j+1} - p_j}{L_j}, \quad b_j := \frac{q_{j+1} - q_j}{L_j}, \quad c_j := p_j - a_j \tilde{L}_{j-1}, \quad d_j := q_j - b_j \tilde{L}_{j-1},$$

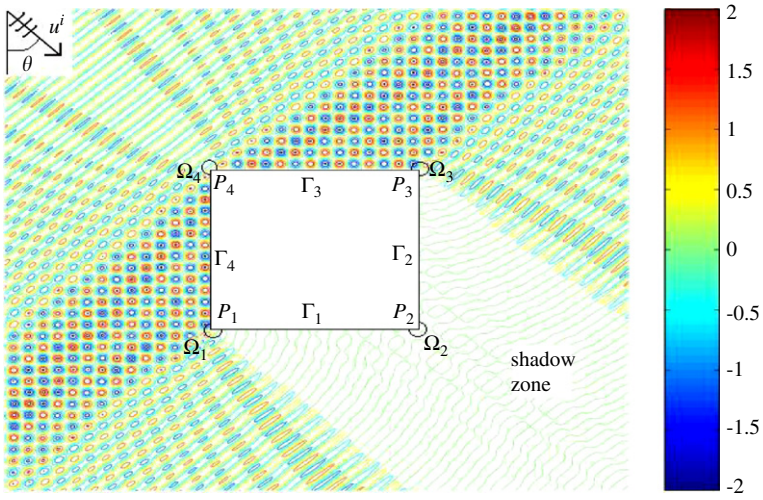

Fig. 1. Total field, scattering by a square of side length 2π, with $k = 10$.

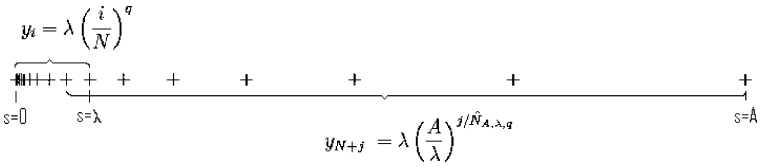

Fig. 2. Composite mesh on $[0, A]$.

where $\tilde{L}_j := \sum_{m=1}^{j} L_m$, and noting that $n_{j1} = b_j$, $n_{j2} = -a_j$, we can rewrite (5) in parametrised form as

$$\phi(s) + \int_0^L K(s,t)\phi(t)\,\mathrm{d}t = f(s), \quad s \in [0, L], \tag{7}$$

where $\phi(s) := (1/k)(\partial u/\partial \mathbf{n})(\mathbf{x}(s))$, $L := \tilde{L}_n$, and for $\mathbf{x}(s) \in \Gamma_l$, $\mathbf{y}(t) \in \Gamma_j$, $l, j = 1, \ldots, n$,

$$K(s,t) := -\frac{1}{2}\left[\eta H_0^{(1)}(kR) + \mathrm{i}k\left[(a_l b_j - b_l a_j)t + b_l(c_l - c_j) - a_l(d_l - d_j)\right]\frac{H_1^{(1)}(kR)}{R}\right], \tag{8}$$

with

$$R = R(s,t) := \sqrt{(a_l s - a_j t + c_l - c_j)^2 + (b_l s - b_j t + d_l - d_j)^2},$$

$$f(s) := 2\mathrm{i}[b_l \sin\theta + a_l \cos\theta + (\eta/k)]\mathrm{e}^{\mathrm{i}k((a_l s + c_l)\sin\theta - (b_l s + d_l)\cos\theta)}. \tag{9}$$

Recalling (6), the first step in the design of our numerical scheme is to separate off the leading order behaviour, namely the contribution from the incident field. For $s \in (\tilde{L}_{l-1}, \tilde{L}_l)$, $l = 1, \ldots, n$, we define $\varphi(s) := \phi(s) - \Psi(s)$, where

$$\Psi(s) := \begin{cases} \dfrac{2}{k}\dfrac{\partial u^i}{\partial \mathbf{n}}(\mathbf{x}(s)) = 2\mathrm{i}(b_l \sin\theta + a_l \cos\theta)\mathrm{e}^{\mathrm{i}k[(a_l s + c_l)\sin\theta - (b_l s + d_l)\cos\theta]}, & l > n_s, \\ 0, & l \leqslant n_s. \end{cases} \tag{10}$$

Substituting into (7) we have

$$\varphi(s) + K\varphi(s) = F(s), \quad s \in [0, L],$$

where

$$K\psi(s) := \int_0^L K(s,t)\psi(t)\,\mathrm{d}t, \quad F(s) := f(s) - \Psi(s) - \int_0^L K(s,t)\Psi(t)\,\mathrm{d}t.$$

This is the integral equation we will solve numerically, with existence and boundedness for $(I + K)^{-1}$ following immediately from [8, Theorem 2.6].

We now define more precisely our approximation space $V_{N,\nu}$. Denoting the wavelength by $\lambda := 2\pi/k$, we begin by defining a graded mesh on a segment $[0, A]$, for $A > \lambda$. We use a composite mesh, with different meshes on the two intervals $[0, \lambda]$ and $[\lambda, A]$, with N and $\hat{N}_{A,\lambda,q}$ points, respectively, in each mesh. For large N, $\hat{N}_{A,\lambda,q}$ is proportional to N. The points on $[0, \lambda]$ accumulate near the origin, and the points on $[\lambda, A]$ become more widely spaced away from λ. The definition of N^* (12) is such that in the case $\hat{N}_{A,\lambda,q} = N^*$, the point to the left of λ lies in both meshes, as shown in Fig. 2.

Definition 1. For $A > \lambda > 0$, $q > 0$, $N = 2, 3, \ldots$, the mesh $\Lambda_{N,A,\lambda,q} := \{y_0, \ldots, y_{N+\hat{N}_{A,\lambda,q}}\}$ consists of the points

$$y_i = \lambda\left(\frac{i}{N}\right)^q, \quad i = 0, \ldots, N,$$

together with the points

$$y_{N+j} := \lambda\left(\frac{A}{\lambda}\right)^{j/\hat{N}_{A,\lambda,q}}, \quad j = 1, \ldots, \hat{N}_{A,\lambda,q}, \tag{11}$$

where $\hat{N}_{A,\lambda,q} = \lceil N^* \rceil$, the smallest integer greater than or equal to N^*, with

$$N^* = \frac{-\log(A/\lambda)}{q\log(1-1/N)}. \tag{12}$$

Assuming $L_j > \lambda$, $j = 1, \ldots, n$, (if this is not the case, we use an appropriate subset of the mesh) we define $q_j := (2\nu+3)/(2\pi/\Omega_j - 1)$, $j = 1, \ldots, n$, and the two meshes

$$\Gamma_j^+ := \tilde{L}_{j-1} + \Lambda_{N,L_j,\lambda,q_j}, \quad \Gamma_j^- := \tilde{L}_j - \Lambda_{N,L_j,\lambda,q_{j+1}}.$$

Letting $\mathrm{e}_\pm(s) := \mathrm{e}^{\pm \mathrm{i}ks}$, $s \in [0, L]$, we then define

$$V_{\Gamma_j^+,\nu} := \{\sigma \mathrm{e}_+ : \sigma \in \Pi_{\Gamma_j^+,\nu}\}, \quad V_{\Gamma_j^-,\nu} := \{\sigma \mathrm{e}_- : \sigma \in \Pi_{\Gamma_j^-,\nu}\},$$

for $j = 1, \ldots, n$, where

$$\Pi_{\Gamma_j^+,\nu} := \{\sigma \in L^2(0,L) : \sigma|_{(\tilde{L}_{j-1}+y_{m-1},\tilde{L}_{j-1}+y_m)} \text{ is a polynomial of degree } \leqslant \nu,$$
$$\text{for } m = 1, \ldots, N + \hat{N}_{L_j,\lambda,q_j}, \text{ and } \sigma|_{(0,\tilde{L}_{j-1})\cup(\tilde{L}_j,L)} = 0\},$$
$$\Pi_{\Gamma_j^-,\nu} := \{\sigma \in L^2(0,L) : \sigma|_{(\tilde{L}_j-\tilde{y}_m,\tilde{L}_j+\tilde{y}_{m-1})} \text{ is a polynomial of degree } \leqslant \nu,$$
$$\text{for } m = 1, \ldots, N + \hat{N}_{L_j,\lambda,q_{j+1}}, \text{ and } \sigma|_{(0,\tilde{L}_{j-1})\cup(\tilde{L}_j,L)} = 0\},$$

with the points of the mesh $\Lambda_{N,L_j,\lambda,q_j}$ given by $y_0, \ldots, y_{N+\hat{N}_{L_j,\lambda,q_j}}$, and the points of the mesh $\Lambda_{N,L_j,\lambda,q_{j+1}}$ given by $\tilde{y}_0, \ldots, \tilde{y}_{N+\hat{N}_{L_j,\lambda,q_{j+1}}}$.

Our approximation space $V_{N,\nu}$ is then the linear span of $\bigcup_{j=1,\ldots,n}\{V_{\Gamma_j^+,\nu} \cup V_{\Gamma_j^-,\nu}\}$. Defining P_{NG} to be the operator of orthogonal projection from L^2 onto $V_{N,\nu}$, a rigorous error analysis [8, Theorem 5.4] demonstrates that

$$\|\varphi - P_{NG}\varphi\|_{2,(0,L)} \leqslant C_\nu \sup_{\mathbf{x}\in D}|u(\mathbf{x})| \frac{n^{1/2}(1+\log^{1/2}(k\max_{j=1,\ldots,n} L_j))}{k^{1/2}N^{\nu+1}}. \tag{13}$$

Moreover defining the Galerkin method approximation $\varphi_{NG} \in V_{N,\nu}$ by

$$(I + P_{NG}K)\varphi_{NG} = P_{NG}F, \tag{14}$$

it is also shown in [8, Theorem 5.3] that

$$\|\varphi - \varphi_{NG}\|_{2,(0,L)} \leqslant C_\nu C_s \sup_{\mathbf{x}\in D}|u(\mathbf{x})| \frac{n^{1/2}(1+\log^{1/2}(k\max_{j=1,\ldots,n} L_j))}{k^{1/2}N^{\nu+1}}. \tag{15}$$

where $C_s := \|(I + P_{NG}K)^{-1}\|_{2,(0,L)}$ is bounded, for N sufficiently large.

Here, instead of projecting orthogonally onto the approximation space we instead use an interpolatory projection. For $p = 1, \ldots, n$, where n is the number of sides of the polygon, we define $n_p^\pm$ to be the number of points of $\Gamma_p^\pm$, so

$$n_p^+ := N + \hat{N}_{L_p,\lambda,q_p}, \quad n_p^- := N + \hat{N}_{L_p,\lambda,q_{p+1}},$$

and we denote the points of $\Gamma_p^\pm$ by $s_{p,l}^\pm$, for $l = 1, \ldots, n_p^\pm$. We define the collocation points to be the midpoints of each interval $(s_{p,j-1}^\pm, s_{p,j}^\pm)$,

$$x_{p,j}^\pm = \frac{s_{p,j}^\pm + s_{p,j-1}^\pm}{2}, \quad j = 1, \ldots, n_p^\pm,$$

and we define P_{NC} to be the interpolatory projection from L^2 onto the approximation space $V_{N,v}$, interpolating at the collocation points. We then solve

$$(I + P_{NC}K)\varphi_{NC} = P_{NC}F. \tag{16}$$

For this scheme we are unable to prove an estimate of the form (15), as we discuss below. However, from (13) we know that the error in the *best* approximation of φ in $V_{N,\Gamma}$ depends only logarithmically on k. Although we cannot guarantee with the collocation scheme that this best approximation will be attained, there exists some hope that a similar estimate might hold when P_{NG} is replaced by P_{NC}.

In order to focus on some of the difficulties involved in the implementation of (16) we consider from now on only the case $v = 0$. Writing φ_N as a linear combination of the basis functions of $V_{N,0}$, we have

$$\varphi_N(s) := \sum_{j=1}^{M_N} c_j \rho_j(s), \tag{17}$$

where ρ_j is the jth basis function and M_N is the dimension of $V_{N,0}$. Denoting the total number of elements supported on $\bigcup_{\mathrm{i}=1}^{p-1} \Gamma_{\mathrm{i}}$ by $\hat{p} := \sum_{i=1}^{p-1} n_i^+ + n_i^-$, we then have for $p = 1, \ldots, n$,

$$\rho_{\hat{p}+j}(s) := \begin{cases} \mathrm{e}^{\mathrm{i}k(s-x^+_{p,j})}\chi_{[s^+_{p,j-1},s^+_{p,j})}(s), & j = 1, \ldots, n_p^+, \\ \mathrm{e}^{-\mathrm{i}k(s-x^-_{p,j})}\chi_{[s^-_{p,j-1},s^-_{p,j})}(s), & j = n_p^+ + 1, \ldots, n_p^+ + n_p^-, \end{cases}$$

where $\chi_{[y_1,y_2)}$ denotes the characteristic function of the interval $[y_1, y_2)$. Substituting (17) into (16) leads to a linear system of the form (where $M_N := \sum_{p=1}^{n} n_p^+ + n_p^-$)

$$\sum_{j=1}^{M_N} c_j[\rho_j(x^\pm_{p,m}) + K\rho_j(x^\pm_{p,m})] = F(x^\pm_{p,m}) \quad \text{for } p = 1, \ldots, n, \;\; m = 1, \ldots, n_p^\pm. \tag{18}$$

Since we have two overlapping meshes, an immediate difficulty presents itself: if $x^+_{p,j} = x^-_{p,m}$ for any $p = 1, \ldots, n$, $j, m = 1, \ldots, n_p^\pm$, i.e. if a collocation point on the mesh Γ_p^+ matches a collocation point on the mesh Γ_p^-, then the system (18) will be singular, and (16) will have no solution. Moreover, if $|x_j^+ - x_m^-| < \varepsilon$, for any j, m, where ε is sufficiently small, i.e. if a collocation point on the mesh Γ_p^+ is too close to a collocation point on the mesh Γ_p^-, then the system may be ill-conditioned.

To avoid this scenario, one approach would be to do away with the overlapping meshes. For example, taking $A = L_j/2$ in the definition of the mesh, putting on each side of the polygon a mesh on $[0, L_j/2]$ and a symmetric mesh on $[L_j/2, L_j]$, and using two basis functions $\mathrm{e}^{\mathrm{i}ks}$ and $\mathrm{e}^{-\mathrm{i}ks}$ on each mesh interval, would allow us to force $|x_j^+ - x_m^-| > \varepsilon$ for any $\varepsilon < \min |y_{j+1} - y_j|$ through an appropriate choice of two collocation points on each interval. However, this approach is unsuitable for two reasons

(1) On the very short intervals near the corners of the polygon, $\mathrm{e}^{\mathrm{i}ks}$ and $\mathrm{e}^{-\mathrm{i}ks}$ will almost match, leading to ill-conditioned systems (see also [9] where a related problem was solved using a mesh of this type).
(2) This approach leads to a much larger number of degrees of freedom than is necessary, with v_- being approximated by far more basis functions than necessary on Γ_j near P_j, and v_+ being approximated by far more basis functions than necessary on Γ_j near P_{j+1}.

For these reasons, we proceed with the overlapping meshes $\Gamma_p^\pm$. For a general polygon, it is hard to say much about the spacing of the collocation points, and hence about the conditioning of the linear system (18). However, considering for simplicity a single side Γ_1, we remark that the collocation points $x^+_{1,j}$ will be very dense on $[0, \lambda]$, and sparse on $(\lambda, L_1]$, whilst the collocation points $x^-_{1,j}$ will be very dense on $[L_1 - \lambda, L_1]$, and sparse on $[0, L_1 - \lambda)$. So, the chance of the system being well conditioned improves if none of the points $x^-_{1,j}$ lie in $[0, \lambda]$ and none of the points $x^+_{1,j}$ lie in $[L_1 - \lambda, L_1]$. For this to be true for the points of $x^+_{1,j}$, we require $s^+_{1,n_p^+-1} < L_1 - 2\lambda$, and recalling (11) this

holds if

$$\hat{N}_{L_1,\lambda,q_0} < \frac{-\log(L_1/\lambda)}{\log(1-2\lambda/L_1)}.$$

Supposing that $L_1 = 2\pi = k\lambda$, so that L_1 is exactly k wavelengths long, where k is the wavenumber, we require $\hat{N}_{L_1,\lambda,q_0} < -\log k/\log(1-2/k)$. Recalling (12) this holds if $N < q_0 k/2$. Since the estimate (13) suggests that N need only grow logarithmically with respect to k as $k \to \infty$ in order to maintain accuracy, this is not a severe restriction. However, for fixed k the restriction $N < q_0 k/2$ suggests that as $N \to \infty$ conditioning problems may result, making the derivation of a conventional asymptotic error estimate rather difficult (see e.g. [4]).

Collocation schemes have been applied very successfully to (1)–(3) in the past (see e.g. [5 (Chapter 8), 10,7,12]), although not particularly with regard to the case that k is large. The success of these schemes suggests that provided the collocation points are sufficiently separated, the scheme should converge in the same manner as the Galerkin scheme. We demonstrate this for our approach via numerical examples in Section 4, relying in our implementation on an examination of the mesh to ensure that there are no conditioning problems.

3. Implementation

The Galerkin approximation (14) leads to a linear system of the form

$$\sum_{j=1}^{N_G} c_j[(\rho_j,\rho_m)+(K\rho_j,\rho_m)] = (f,\rho_m), \quad \text{for } m = 1,2,\ldots,N_G.$$

Recalling (8), (9), this leaves many double integrals of the form

$$(K\rho_j,\rho_m) = \int_{\mathrm{supp}\rho_m}\int_{\mathrm{supp}\rho_j}\left(\frac{\partial\Phi}{\partial\mathbf{n}}+\mathrm{i}\eta\Phi\right)\rho_j(s)\rho_m(t)\,\mathrm{d}s\,\mathrm{d}t, \tag{19}$$

to evaluate (see [8,18] for details). This is a double integral over the support of each of the basis functions of an oscillatory function, since the term $(\partial\Phi/\partial\mathbf{n}+\mathrm{i}\eta\Phi)$ is oscillatory as are the basis functions ρ_j and ρ_m. As described in [16], in principal at least an integral should become easier to evaluate as it becomes more oscillatory, as due to cancellation of oscillating terms the exact value will tend to zero more quickly as the oscillations increase. However, using this information to construct an accurate numerical scheme for highly oscillatory integrals of the form (19) is a difficult task, and most schemes presented recently in the literature for the evaluation of highly oscillatory integrals focus on one-dimensional integrals.

However, for the linear system (18) the single integrals

$$K\rho_j(s_m) = \int_{y_j}^{y_{j+1}} K(s_m,t)\mathrm{e}^{\pm\mathrm{i}k(t-s_j)}\,\mathrm{d}t, \tag{20}$$

are a little easier to evaluate, where here $s_m, m = 1,\ldots,M_N$ represent the collocation points and $[y_j, y_{j+1}]$ the support of ρ_j.

If the collocation point and the support of the basis function lie on the same side of the polygon,

$$K(s_m,t) = -\frac{\eta}{4}H_0^{(1)}(k|s_m-t|),$$

and using the identity [20, Eq. (12.31)]

$$H_0^{(1)}(s) = -\frac{2\mathrm{i}}{\pi}\int_0^\infty \frac{\mathrm{e}^{(\mathrm{i}-t)s}}{t^{1/2}(t-2\mathrm{i})^{1/2}}\,\mathrm{d}t, \quad s>0,$$

we can write (20) as

$$\frac{\mathrm{i}\eta}{2\pi\mathrm{e}^{\mathrm{i}ks_j}}\int_0^\infty \frac{I(r)}{r^{1/2}(r-2\mathrm{i})^{1/2}}\,\mathrm{d}r,$$

where

$$I(r) := \int_{y_j}^{y_{j+1}} \mathrm{e}^{(\mathrm{i}-r)k|s_m-t|+\sigma_j \mathrm{i}kt}\,\mathrm{d}t, \tag{21}$$

with $\sigma_j = \pm 1$. It is shown in [3] that

$$I(r) = \begin{cases} \dfrac{\mathrm{e}^{k(r-\mathrm{i})s_m}\left(\mathrm{e}^{-ky_j(r-\mathrm{i}(1+\sigma_j))} - \mathrm{e}^{-ky_{j+1}(r-\mathrm{i}(1+\sigma_j))}\right)}{k\left(r-\mathrm{i}(1+\sigma_j)\right)}, & s_m < y_j, \\ \dfrac{\mathrm{e}^{-k(r-i)s_m}\left(-\mathrm{e}^{ky_j(r+\mathrm{i}(\sigma_j-1))} + \mathrm{e}^{ky_{j+1}(r+\mathrm{i}(\sigma_j-1))}\right)}{k(r+\mathrm{i}(\sigma_j-1))}, & s_m > y_{j+1}, \\ \dfrac{\mathrm{e}^{k\mathrm{i}s_m\sigma_j} - \mathrm{e}^{rk(y_j-s_m)+\mathrm{i}k(s_m+y_j(\sigma_j-1))}}{\mathrm{i}k((\sigma_j-1))} & \\ \quad + \dfrac{\mathrm{e}^{\mathrm{i}ks_m\sigma_j} - \mathrm{e}^{rk(s_m-y_{j+1})+\mathrm{i}k(y_{j+1}(1+\sigma_j)-s_m)}}{\mathrm{i}k(r-(1+\sigma_j))}, & y_j < s_m < y_{j+1}, \end{cases}$$

and then to evaluate (21) we make the substitution $r = s^2/(1-s^2)$, to reduce the interval of integration to $[0, 1]$ and eliminate the singularity at $r = 0$, allowing us to use standard Gaussian quadrature, as the remaining integral is not oscillatory.

The second and more difficult case we need to consider is that where the collocation point and the support of the basis function lie on different sides of the polygon. In this case, we must evaluate integrals of the form

$$J := \int_a^b \left[H_0^1(k\sqrt{s^2+c^2}) + \frac{\mathrm{i}sH_1^1(k\sqrt{s^2+c^2})}{\sqrt{s^2+c^2}}\right]\mathrm{e}^{\pm \mathrm{i}ks}\,\mathrm{d}s,$$

where $a, b, c \in \mathbb{R}$. Defining

$$G(s) := \left[H_0^1(k\sqrt{s^2+c^2}) + \frac{\mathrm{i}sH_1^1(k\sqrt{s^2+c^2})}{\sqrt{s^2+c^2}}\right]\mathrm{e}^{-\mathrm{i}k\sqrt{s^2+c^2}},$$

it follows from standard properties of Hankel functions (see e.g. [2]) that $G(s)$ is slowly oscillating compared to $\mathrm{e}^{\mathrm{i}k(\sqrt{s^2+c^2}\pm s)}$. We thus consider evaluation of

$$J_+ := \int_a^b G(s)\mathrm{e}^{\mathrm{i}k(s+\sqrt{s^2+c^2})}\,\mathrm{d}s,$$

with the method for the evaluation of $J_- := \int_a^b G(s)\mathrm{e}^{\mathrm{i}k(-s+\sqrt{s^2+c^2})}\,\mathrm{d}s$ following analogously. Making the substitution $t = s + \sqrt{s^2+c^2}$ we have

$$J_+ = \int_{a+\sqrt{a^2+c^2}}^{b+\sqrt{b^2+c^2}} G\left(\frac{t^2-c^2}{2t}\right)\frac{\sqrt{t^2+c^2}}{2t^2}\mathrm{e}^{\mathrm{i}kt}\,\mathrm{d}t,$$

and methods for evaluating this type of integral are well established [16].

We remark that the evaluation of $\int_{L_{n_s}}^{L} K(s_m, t)(\partial u^i/\partial \mathbf{n})(t)\mathrm{d}t$ on the right-hand side of (18) is carried out by a combination of the above two procedures.

4. Numerical results

For the Galerkin method described in [8] we have the error estimate (15). Although no such estimate has been proved for the collocation scheme described here, we hope to demonstrate via numerical examples that a similar result might be applicable.

Table 1
Relative L_2 errors, $k = 10, 20, 40, 80, 160$, $N = 4, 8, 16, 32, 64$

k	N	M_N	$\|\varphi - \varphi_{NC}\|_2/\|\varphi\|_2$	EOC
10	4	48	4.7335×10^{-1}	0.8
	8	96	2.6980×10^{-1}	1.0
	16	192	1.2670×10^{-1}	0.9
	32	376	6.8440×10^{-2}	1.0
	64	752	3.3034×10^{-2}	
20	4	48	7.1085×10^{-1}	1.2
	8	104	3.0762×10^{-1}	1.0
	16	200	1.7872×10^{-1}	1.2
	32	392	5.5728×10^{-2}	1.0
	64	792	4.1295×10^{-2}	
40	4	56	5.4597×10^{-1}	0.7
	8	104	3.4089×10^{-1}	0.3
	16	208	3.6095×10^{-1}	0.3
	32	416	2.8317×10^{-1}	1.0
	64	824	3.7158×10^{-2}	
80	4	56	4.6096×10^{-1}	1.0
	8	112	2.3333×10^{-1}	0.8
	16	216	1.5975×10^{-1}	0.6
	32	432	1.4203×10^{-1}	0.9
	64	864	4.4374×10^{-2}	
160	4	56	4.4455×10^{-1}	−0.1
	8	112	4.6445×10^{-1}	0.5
	16	224	2.3456×10^{-1}	0.7
	32	456	9.3327×10^{-2}	0.8
	64	904	4.8153×10^{-2}	

As a numerical example we consider the problem of scattering by a square of side length 2π, with the angle of incidence $\pi/4$ as measured anticlockwise from the downward vertical. In calculating the errors we need an "exact" solution, and this is computed using the Galerkin scheme (for which we have proved convergence) using a large number of degrees of freedom. We remark that our test problem is the same as that considered in [8].

Table 1 demonstrates the results obtained using the collocation method for increasing values of k and N. For each k, we show the values of N, the total number of degrees of freedom M_N, the relative error $\|\varphi - \varphi_{NC}\|_2/\|\varphi\|_2$ and the estimated order of convergence

$$\mathrm{EOC} := -\frac{1}{M}\sum_{j=1}^{M} \log_2 \frac{\|\varphi - \varphi_{2^j NC}\|_2}{\|\varphi - \varphi_{2^{j-1} NC}\|_2}.$$

The results appear to suggest that for each value of k the solution is converging to the same solution as that achieved by the Galerkin scheme, for which we have proved convergence to the true solution of the integral equation, and at roughly the same rate (i.e. EOC ≈ 1, as it would be if the estimate (15) held for the collocation scheme as well). Moreover, the relative error remains roughly constant for fixed N as k increases, suggesting that, as for the Galerkin scheme, the number of degrees of freedom required to achieve a prescribed level of accuracy grows only logarithmically with respect to the frequency. Further numerical results can be found in [3].

5. Conclusions

We have proposed and implemented a new collocation method for solving problems of high-frequency scattering by convex polygons. We use the same approximation space as for the Galerkin method in [8], and our numerical results appear to suggest that we achieve the same convergence rate, namely that the number of degrees of freedom required

to achieve a prescribed level of accuracy grows only logarithmically with respect to the frequency. Moreover, the collocation method exhibits a significant reduction in the computational time compared to the Galerkin scheme.

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